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JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

Journal of Computational and Applied Mathematics 222 (2008) 440–457

www.elsevier.com/locate/cam

Synchronization of neural networks based on parameter identification and via output or state coupling

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Received 22 August 2006; received in revised form 13 November 2007

Abstract

For neural networks with all the parameters unknown, we focus on the global robust synchronization between two coupled neural networks with time-varying delay that are linearly and unidirectionally coupled. First, we use Lyapunov functionals to establish general theoretical conditions for designing the coupling matrix. Neither symmetry nor negative (positive) definiteness of the coupling matrix are required; under less restrictive conditions, the two coupled chaotic neural networks can achieve global robust synchronization regardless of their initial states. Second, by employing the invariance principle of functional differential equations, a simple, analytical, and rigorous adaptive feedback scheme is proposed for the robust synchronization of almost all kinds of coupled neural networks with time-varying delay based on the parameter identification of uncertain delayed neural networks. Finally, numerical simulations validate the effectiveness and feasibility of the proposed technique.

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Keywords: Global robust synchronization; Neural networks; Lyapunov functional; Parameter identification; Output coupling; State coupling

1. Introduction

In the past few years, there has been increasing interest in the potential applications of the dynamics of artificial neural networks in many areas [1–16]. In such applications, analysis of the equilibrium points is a prerequisite. Thus, different types of neural networks with or without time delays have been widely investigated and many stability criteria have been obtained [2–7,9–16].

In 1990, Pecora and Carroll [17] addressed the synchronization of chaotic systems using a drive–response conception. The idea is to use the output of the drive system to control the response system so that they oscillate in a synchronized manner. Research on the synchronization of chaotic activity has broadened considerably in the last few years. Besides the original master–slave mechanism for chaos synchronization, a wide variety of approaches have been presented for the synchronization of chaotic systems which include linear feedback control [18,24], nonlinear feedback control [19], impulsive control method [20], and adaptive design control [21–23], among many others. Synchronization in chaotic systems has been utilized in many applications. It was used to understand self-organization

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behavior in the brain as well as in ecological systems [25,26], and has been applied to secure communications [22], among others.

Artificial neural network models can exhibit chaotic behavior [31–34], and so, synchronization of chaotic neural networks has also become an important area of study. Nowadays, some authors pay attention to the synchronization of neural networks or complex networks [35–40]. Synchronization of coupled delayed neural networks and applications to chaotic cellular neural networks in [35] have resulted in a theoretical condition for synchronization under the assumption that the coupling matrix is irreducible. In [36], some new delay-dependent conditions for a general complex dynamical network model with coupling delays were presented. Cao et al. [37] reported a simple adaptive feedback scheme for the synchronization of coupled uncertain neural networks with or without time-varying delay based on the invariant principles of functional differential equations.

However, most of the above studies are valid only for the chaotic neural networks whose parameters are precisely known. But in a practical situation, the parameters of some systems cannot be exactly known a priori, the effect of these uncertainties will destroy the synchronization and even break it [27–30]. Therefore, it is essential to investigate the synchronization of delayed chaotic systems in the presence of unknown parameters.

In this paper, we focus on synchronization dynamics of recurrently delayed neural networks with all the parameters unknown and consider more general coupling conditions, including outputs or states that result in different theoretical synchronization criteria. Via some novel approaches of parameter identification and the invariance principle of functional differential equations, it is shown that one can rapidly achieve global robust synchronization of such networks while identifying all the unknown parameters dynamically. In addition, it is quite robust against the effect of noise and able to respond rapidly to changes in tracking parameters of the master system.

This paper is organized as follows. After giving some preliminaries, the problem formulations are presented for synchronization and parameter identification of coupled neural networks with time-varying delay. Section 3 deals with the robust synchronization problem of such uncertain neural networks via output coupling; some criteria are derived for determining the global robust synchronization based on the parameter identification of the uncertain delayed neural networks. Then, the robust synchronization for the two coupled neural networks is studied via state coupling in Section 4; a simple, practical, and rigorous adaptive feedback controlling law is proposed for the robust synchronization, where the approaches are based on the invariance principle of functional differential equations. In Section 5, numerical simulations show the effectiveness and feasibility of the proposed technique. Finally, some conclusions are given in Section 5.

Notations. In the sequel, we denote A^T and A^{-1} the transpose and the inverse of any square matrix A . We use $A > 0$ ($A < 0$) to denote a positive- (negative-) definite matrix A ; and I is used to denote the $n \times n$ identity matrix. $\|A\|$ denotes the spectral norm of matrix A . Let \mathbf{R} denote the set of real numbers, \mathbf{R}^n denotes the n -dimensional Euclidean space and $\mathbf{R}^{n \times m}$ the set of all $n \times m$ real matrices. $\text{diag}(\cdot)$ denotes a block diagonal matrix. $\lambda_{\max}(\cdot)$ or $\lambda_{\min}(\cdot)$ denotes the largest or smallest eigenvalue of a matrix, respectively.

2. Formulation of synchronization in neural networks

First, we consider the following neural network models in a general form, allowing for networks with or without delay:

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} f_j(x_j(t - \tau(t))) + J_i, \quad i = 1, 2, \dots, n \quad (1)$$

or, in a compact form,

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) + J, \quad (2)$$

where $x(t) = (x_1(t), \dots, x_n(t))^T \in \mathbf{R}^n$ is the state vector of the neural network; $C = \text{diag}(c_1, \dots, c_n)$ is a diagonal matrix with $c_i > 0$, $i = 1, 2, \dots, n$, $A = (a_{ij})_{n \times n}$ is a weight matrix; $B = (b_{ij})_{n \times n}$ is the delayed weight matrix; $J = (J_1, \dots, J_n)^T \in \mathbf{R}^n$ is the input vector function; $\tau(t) \geq 0$ is the transmission delay; $f(x(t)) = [f_1(x_1(t)), \dots, f_n(x_n(t))]^T$.

Clearly, most common existing neural network models can be represented in (1) or (2).

Throughout the paper, we have the following two assumptions:

(A1) Each $f_j : \mathbf{R} \rightarrow \mathbf{R}$ is monotonic nondecreasing and globally Lipschitz, i.e. there exist positive scalars $k_j > 0$ such that

$$0 \leq \frac{f_j(x) - f_j(y)}{x - y} \leq k_j, \quad j = 1, 2, \dots, n$$

for any $x, y \in \mathbf{R}, x \neq y$. Obviously, from the well-known Hopfield neural network and cellular neural network belong to this type. Moreover, the requirement for f_j is less restrictive than the Sigmoidal function and the piecewise linear function $\frac{1}{2}(|x + 1| - |x - 1|)$.

(A2) $\tau(t) \geq 0$ is a differential function with $\tau^* = \max_t(\tau(t))$ and $0 \leq \dot{\tau}(t) \leq \sigma < 1$ for all t .

The initial conditions of (1) are given by $x_i(t) = \phi_i(t) \in \mathcal{C}([-\tau^*, 0], \mathbf{R})$, where $\mathcal{C}([-\tau^*, 0], \mathbf{R})$ denotes the set of all continuous functions from $[-\tau^*, 0]$ to \mathbf{R} .

Now consider the master (or drive) system in the form of the delayed neural networks (1) or (2), which may be a chaotic system. We also introduce an auxiliary variable $y(t) = (y_1(t), \dots, y_n(t))^T \in \mathbf{R}^n$, the slave (or response) system is given by the following equation:

$$\dot{y}(t) = -\tilde{C}y(t) + \tilde{A}f(y(t)) + \tilde{B}f(y(t - \tau(t))) + J, \quad (3)$$

which has the same structure as the master system but all the parameters $\tilde{C} = \text{diag}(\tilde{c}_1, \dots, \tilde{c}_n)$, $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$, $\tilde{B} = (\tilde{b}_{ij})_{n \times n}$ are completely unknown, or uncertain. The initial conditions of (3) are given by $y_i(t) = \psi_i(t) \in \mathcal{C}([-\tau^*, 0], \mathbf{R})$ ($i = 1, 2, \dots, n$), where $\mathcal{C}([-\tau^*, 0], \mathbf{R})$ denotes the set of all continuous functions from $[-\tau^*, 0]$ to \mathbf{R} . In a practical situation, the output signals of the master system (2) can be received by the slave system (3), but the parameter vector of the master system (2) may not be known a priori, even waits for identifying. To estimate all the unknown parameters, by adding the controller to the slave system (3), we have the following controlled slave system:

$$\dot{y}(t) = -\tilde{C}y(t) + \tilde{A}f(y(t)) + \tilde{B}f(y(t - \tau(t))) + J + U(t), \quad (4)$$

where $U(t) = [u_1, \dots, u_n]^T$ is the driving signal which can take different forms. Therefore, the goal of control is to design and implement an appropriate controller $U(t)$ for the slave system and parameters' adaptive estimation laws of \tilde{C} , \tilde{A} and \tilde{B} such that the controlled slave system could be synchronous with the master system (2), and all the parameters $\tilde{C} \rightarrow C$, $\tilde{A} \rightarrow A$ and $\tilde{B} \rightarrow B$ as $t \rightarrow \infty$.

Next, we introduce a lemma, which is needed in the proof of the main theorem.

Lemma 1 (Xu et al. [41]). Let $\Sigma_1, \Sigma_2, \Sigma_3$ be real matrices of appropriate dimensions with $\Sigma_3 > 0$. Then for any vectors x and y with appropriate dimensions,

$$2x^T \Sigma_1^T \Sigma_2 y \leq x^T \Sigma_1^T \Sigma_3 \Sigma_1 x + y^T \Sigma_2^T \Sigma_3^{-1} \Sigma_2 y.$$

3. Synchronization of neural networks via output coupling

Based on Lyapunov functionals and estimation techniques, we can provide design rules for the controller gain matrix Ω , with which global robust synchronization is ensured.

Theorem 1. Under the assumptions (A1) and (A2), let the controller $u_i(t) = \sum_{j=1}^n w_{ij} (f_j(y_j(t)) - f_j(x_j(t)))$ and the parameters' adaptive laws of $\tilde{C} = \text{diag}(\tilde{c}_1, \dots, \tilde{c}_n)$, $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ and $\tilde{B} = (\tilde{b}_{ij})_{n \times n}$ are chosen as below:

$$\begin{cases} \dot{\tilde{c}}_i = \gamma_i e_i(t) y_i(t), & i = 1, 2, \dots, n, \\ \dot{\tilde{a}}_{ij} = -\alpha_{ij} e_i(t) f_j(y_j(t)), & i, j = 1, 2, \dots, n, \\ \dot{\tilde{b}}_{ij} = -\beta_{ij} e_i(t) f_j(y_j(t - \tau(t))), & i, j = 1, 2, \dots, n, \end{cases} \quad (5)$$

in which $e_i(t) = y_i(t) - x_i(t)$, $\gamma_i > 0$ ($i = 1, 2, \dots, n$) and $\alpha_{ij} > 0$, $\beta_{ij} > 0$, ($i, j = 1, 2, \dots, n$) are arbitrary constants. If, provided that for controller gain matrix $\Omega = (w_{ij})_{n \times n}$, the following condition

$$\lambda_{\max}(-C) + \lambda_{\max} \left(\frac{1}{2} (A + \Omega)(A + \Omega)^T \right) + \lambda_{\max} \left(\frac{1}{2} B B^T \right) + \frac{1}{2} k + \frac{1}{2(1 - \sigma)} k < 0 \quad (6)$$

is satisfied, where $k = \max_{1 \leq i \leq n} k_i^2$, then the controlled uncertain slave system (3) will globally synchronize with the master system (2). Moreover,

$$\lim_{t \rightarrow \infty} (\tilde{c}_i - c_i) = \lim_{t \rightarrow \infty} (\tilde{a}_{ij} - a_{ij}) = \lim_{t \rightarrow \infty} (\tilde{b}_{ij} - b_{ij}) = 0,$$

for all $i, j = 1, 2, \dots, n$.

Proof. When the two neural networks are coupled via outputs, substituting $U(t) = \Omega(f(y(t)) - f(x(t)))$ into (3), and let $e(t) = y(t) - x(t)$ be the synchronization error between the master system (2) and the controlled slave system (3), one can get the error dynamical system as follows:

$$\begin{aligned} \dot{e}_i(t) = & -c_i e_i(t) + \sum_{j=1}^n (a_{ij} + w_{ij}) g_j(e_j(t)) + \sum_{j=1}^n b_{ij} g_j(e_j(t - \tau(t))) - (\tilde{c}_i - c_i) y_i(t) \\ & + \sum_{j=1}^n (\tilde{a}_{ij} - a_{ij}) f_j(y_j(t)) + \sum_{j=1}^n (\tilde{b}_{ij} - b_{ij}) f_j(y_j(t - \tau(t))), \end{aligned} \quad (7)$$

or

$$\begin{aligned} \dot{e}(t) = & -Ce(t) + (A + \Omega)g(e(t)) + Bg(e(t - \tau(t))) \\ & - (\tilde{C} - C)y(t) + (\tilde{A} - A)f(y(t)) + (\tilde{B} - B)f(y(t - \tau(t))), \end{aligned} \quad (8)$$

where

$e(t) = (e_1(t), \dots, e_n(t))^T$, $f(y(t)) = [f_1(y_1(t)), \dots, f_n(y_n(t))]^T$ and $g(e(t)) = [g_1(e_1(t)), \dots, g_n(e_n(t))]^T$ with $g_i(e_i(t)) = f_i(e_i(t) + x_i(t)) - f_i(x_i(t))$, $i = 1, 2, \dots, n$.

According to the properties of (A1), $g_i(\cdot)$ possess the following properties:

$$|g_i(e_i)| \leq k_i |e_i|, \quad 0 \leq e_i g_i(e_i) \leq k_i e_i^2, \quad i = 1, 2, \dots, n. \quad (9)$$

Now construct a Lyapunov functional of the form

$$\begin{aligned} V(t) = & \frac{1}{2} \left\{ e^T(t) e(t) + \frac{1}{1 - \sigma} \int_{t-\tau(t)}^t g^T(e(s)) g(e(s)) ds \right. \\ & \left. + \sum_{i=1}^n \left[\frac{1}{\gamma_i} (\tilde{c}_i - c_i)^2 + \sum_{j=1}^n \frac{1}{\alpha_{ij}} (\tilde{a}_{ij} - a_{ij})^2 + \sum_{j=1}^n \frac{1}{\beta_{ij}} (\tilde{b}_{ij} - b_{ij})^2 \right] \right\}. \end{aligned} \quad (10)$$

Differentiating V with respect to time along the solution of (8), and by using Lemma 1, we have

$$\begin{aligned} \dot{V}(t) \leq & -e^T(t) C e(t) + \frac{1}{2} e^T(t) (A + \Omega)(A + \Omega)^T e(t) + \frac{1}{2} g^T(e(t)) g(e(t)) \\ & + \frac{1}{2} e^T(t) B B^T e(t) + \frac{1}{2} g^T(e(t - \tau(t))) g(e(t - \tau(t))) \\ & + \frac{1}{2(1 - \sigma)} g^T(e(t)) g(e(t)) - \frac{1}{2} g^T(e(t - \tau(t))) g(e(t - \tau(t))) \\ = & -e^T(t) C e(t) + \frac{1}{2} e^T(t) (A + \Omega)(A + \Omega)^T e(t) \\ & + \frac{1}{2} g^T(e(t)) g(e(t)) + \frac{1}{2} e^T(t) B B^T e(t) + \frac{1}{2(1 - \sigma)} g^T(e(t)) g(e(t)) \\ \leq & e^T(t) \left[\lambda_{\max}(-C) + \lambda_{\max} \left(\frac{1}{2} (A + \Omega)(A + \Omega)^T \right) + \lambda_{\max} \left(\frac{1}{2} B B^T \right) + \frac{1}{2} k + \frac{1}{2(1 - \sigma)} k \right] e(t). \end{aligned} \quad (11)$$

If the condition (6) of Theorem 1 is satisfied, from (11), we have $\dot{V}(t) = 0$ if and only if $e(t) = 0$. According to the well-known invariant principle of functional differential equations, the orbit of system (8), starting with arbitrary initial value, converges asymptotically to the largest invariant set E contained in $\dot{V}(t) = 0$ as $t \rightarrow \infty$, where the set $E = \{e(t) = 0 | \tilde{C} = C, \tilde{A} = A, \tilde{B} = B, \epsilon = \epsilon_0 \in \mathbf{R}^n\}$. The unknown parameters \tilde{C} , \tilde{A} and \tilde{B} with arbitrary

initial values will approximate asymptotically the parameter identification values C , A and B of the drive system (2), respectively. This complete the proof. \square

Theorem 2. Under the assumptions (A1) and (A2), let the controller $u_i(t) = \sum_{j=1}^n w_{ij} (f_j(y_j(t)) - f_j(x_j(t)))$ and the parameters' adaptive laws of \tilde{C} , \tilde{A} and \tilde{B} are chosen as below:

$$\begin{cases} \dot{\tilde{c}}_i = \gamma_i e_i(t) y_i(t), & i = 1, 2, \dots, n, \\ \dot{\tilde{a}}_{ij} = -\alpha_{ij} e_i(t) f_j(y_j(t)), & i, j = 1, 2, \dots, n, \\ \dot{\tilde{b}}_{ij} = -\beta_{ij} e_i(t) f_j(y_j(t - \tau(t))), & i, j = 1, 2, \dots, n, \end{cases} \quad (12)$$

in which $\gamma_i > 0$ ($i = 1, 2, \dots, n$) and $\alpha_{ij} > 0$, $\beta_{ij} > 0$, ($i, j = 1, 2, \dots, n$) are arbitrary constants. If, provided that for controller gain matrix $\Omega = (w_{ij})_{n \times n}$, there exist n positive constants $p_i > 0$, $i = 1, 2, \dots, n$, such that

$$\begin{aligned} & -p_i c_i + p_i (a_{ii} + w_{ii}) k_i + \sum_{j=1, j \neq i}^n p_i |a_{ij} + w_{ij}| k_j \\ & + \sum_{j=1, j \neq i}^n p_j |a_{ji} + w_{ji}| k_i + \sum_{j=1}^n p_i |b_{ij}| k_j + \frac{1}{1-\sigma} \sum_{j=1}^n p_i |b_{ji}| k_i < 0, \end{aligned} \quad (13)$$

then the controlled uncertain slave system (3) will globally synchronize with the master system (2). Moreover,

$$\lim_{t \rightarrow \infty} (\tilde{c}_i - c_i) = \lim_{t \rightarrow \infty} (\tilde{a}_{ij} - a_{ij}) = \lim_{t \rightarrow \infty} (\tilde{b}_{ij} - b_{ij}) = 0,$$

for all $i, j = 1, 2, \dots, n$.

Proof. Consider another Lyapunov functional of the form

$$\begin{aligned} V(t) = & \frac{1}{2} \sum_{i=1}^n p_i e_i^2(t) + \frac{1}{2(1-\sigma)} \sum_{i=1}^n \sum_{j=1}^n p_i |b_{ji}| \int_{t-\tau(t)}^t |e_i(s)| |g_i(e_i(s))| ds \\ & + \frac{1}{2} \sum_{i=1}^n \left[\frac{p_i}{\gamma_i} (\tilde{c}_i - c_i)^2 + \sum_{j=1}^n \frac{p_i}{\alpha_{ij}} (\tilde{a}_{ij} - a_{ij})^2 + \sum_{j=1}^n \frac{p_i}{\beta_{ij}} (\tilde{b}_{ij} - b_{ij})^2 \right]. \end{aligned} \quad (14)$$

Calculating the derivative of $V(t)$ along the trajectories of the error system (7), we obtain that

$$\begin{aligned} \dot{V}(t) = & \sum_{i=1}^n p_i e_i(t) \left(-c_i e_i(t) + \sum_{j=1}^n (a_{ij} + w_{ij}) g_j(e_j(t)) + \sum_{j=1}^n b_{ij} g_j(e_j(t - \tau(t))) - (\tilde{c}_i - c_i) y_i(t) \right. \\ & \left. + \sum_{j=1}^n (\tilde{a}_{ij} - a_{ij}) f_j(y_j(t)) + \sum_{j=1}^n (\tilde{b}_{ij} - b_{ij}) f_j(y_j(t - \tau(t))) \right) \\ & + \frac{1}{2(1-\sigma)} \sum_{i=1}^n \sum_{j=1}^n p_i |b_{ji}| |e_i(t)| |g_i(e_i(t))| \\ & - \frac{1-\dot{\tau}(t)}{2(1-\sigma)} \sum_{i=1}^n \sum_{j=1}^n p_i |b_{ji}| |e_i(t - \tau(t))| |g_i(e_i(t - \tau(t)))| \\ & + \sum_{i=1}^n \left[\frac{p_i}{\gamma_i} (\tilde{c}_i - c_i) \dot{\tilde{c}}_i + \sum_{j=1}^n \frac{p_i}{\alpha_{ij}} (\tilde{a}_{ij} - a_{ij}) \dot{\tilde{a}}_{ij} + \sum_{j=1}^n \frac{p_i}{\beta_{ij}} (\tilde{b}_{ij} - b_{ij}) \dot{\tilde{b}}_{ij} \right] \\ \leq & \sum_{i=1}^n \left\{ -p_i c_i e_i^2(t) + p_i (a_{ii} + w_{ii}) |e_i(t)| |g_i(e_i(t))| \right\} \\ & + \sum_{i=1}^n \sum_{j=1, j \neq i}^n p_i |a_{ij} + w_{ij}| |e_i(t)| |g_j(e_j(t))| + \sum_{i=1}^n \sum_{j=1}^n p_i |b_{ij}| |e_i(t)| |g_j(e_j(t - \tau(t)))| \end{aligned}$$

$$+ \frac{1}{2(1-\sigma)} \sum_{i=1}^n \sum_{j=1}^n p_i |b_{ji}| |e_i(t)| |g_i(e_i(t))| - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n p_i |b_{ji}| |e_i(t - \tau(t))| |g_i(e_i(t - \tau(t)))|. \quad (15)$$

In the above estimations, using the equality $2ab \leq a^2 + b^2$, one can obtain

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1, j \neq i}^n p_i |a_{ij} + w_{ij}| |e_i(t)| |g_j(e_j(t))| &= \sum_{i=1}^n \sum_{j=1, j \neq i}^n p_i |a_{ij} + w_{ij}| k_j (|e_i(t)|) (|g_j(e_j(t))|/k_j) \\ &\leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1, j \neq i}^n p_i |a_{ij} + w_{ij}| k_j e_i^2(t) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1, j \neq i}^n p_i |a_{ij} + w_{ij}| g_j^2(e_j(t))/k_j. \end{aligned} \quad (16)$$

The second term in (17) can be further relaxed

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1, j \neq i}^n p_i |a_{ij} + w_{ij}| g_j^2(e_j(t))/k_j &\leq \sum_{i=1}^n \sum_{j=1, j \neq i}^n p_i |a_{ij} + w_{ij}| |g_j(e_j(t))| |e_j(t)| \\ &= \sum_{i=1}^n \sum_{j=1, j \neq i}^n p_j |a_{ji} + w_{ji}| |g_i(e_i(t))| |e_i(t)|. \end{aligned} \quad (17)$$

Similarly, we have

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n p_i |b_{ij}| |e_i(t)| |g_j(e_j(t - \tau(t)))| \\ \leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n p_i |b_{ij}| k_j e_i^2(t) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n p_j |b_{ji}| |g_i(e_i(t - \tau(t)))| |e_i(t - \tau(t))|. \end{aligned} \quad (18)$$

Combining (16)–(19), we get

$$\begin{aligned} \dot{V}(t) &\leq \sum_{i=1}^n \left\{ -p_i c_i e_i^2(t) + p_i (a_{ii} + w_{ii}) |e_i(t)| |g_i(e_i(t))| \right\} + \sum_{i=1}^n \sum_{j=1, j \neq i}^n p_i |a_{ij} + w_{ij}| k_j e_i^2(t) \\ &\quad + \sum_{i=1}^n \sum_{j=1, j \neq i}^n p_j |a_{ji} + w_{ji}| |g_i(e_i(t))| |e_i(t)| + \sum_{i=1}^n \sum_{j=1}^n p_i |b_{ij}| k_j e_i^2(t) \\ &\quad + \frac{1}{1-\sigma} \sum_{i=1}^n \sum_{j=1}^n p_i |b_{ji}| |e_i(t)| |g_i(e_i(t))| \\ &\leq \sum_{i=1}^n \left\{ -p_i c_i + p_i (a_{ii} + w_{ii}) k_i + \sum_{j=1, j \neq i}^n p_i |a_{ij} + w_{ij}| k_j \right. \\ &\quad \left. + \sum_{j=1, j \neq i}^n p_j |a_{ji} + w_{ji}| k_i + \sum_{j=1}^n p_i |b_{ij}| k_j + \frac{1}{1-\sigma} \sum_{j=1}^n p_i |b_{ji}| k_i \right\} e_i^2(t). \end{aligned} \quad (19)$$

So, if the condition (13) of Theorem 2 is satisfied, from (20), we have $\dot{V}(t) = 0$ if and only if $e(t) = 0$. According to the well-known invariant principle of functional differential equations, the orbit of system (8), starting with arbitrary initial value, converges asymptotically to the largest invariant set E contained in $\dot{V}(t) = 0$ as $t \rightarrow \infty$, where the set $E = \{e(t) = 0 | \tilde{C} = C, \tilde{A} = A, \tilde{B} = B, \epsilon = \epsilon_0 \in \mathbf{R}^n\}$. This complete the proof. \square

Theorem 3. Under the assumptions (A1) and (A2), let the controller $u_i(t) = \sum_{j=1}^n w_{ij} (f_j(y_j(t)) - f_j(x_j(t)))$ and the parameters' adaptive laws of \tilde{C} , \tilde{A} and \tilde{B} are chosen as below:

$$\begin{cases} \dot{\tilde{c}}_i = \gamma_i g_i(e_i(t)) y_i(t), & i = 1, 2, \dots, n, \\ \dot{\tilde{a}}_{ij} = -\alpha_{ij} g_i(e_i(t)) f_j(y_j(t)), & i, j = 1, 2, \dots, n, \\ \dot{\tilde{b}}_{ij} = -\beta_{ij} g_i(e_i(t)) f_j(y_j(t - \tau(t))), & i, j = 1, 2, \dots, n, \end{cases} \quad (20)$$

in which $\gamma_i > 0$ ($i = 1, 2, \dots, n$) and $\alpha_{ij} > 0$, $\beta_{ij} > 0$, ($i, j = 1, 2, \dots, n$) are arbitrary constants. If, provided that for controller gain matrix $\Omega = (w_{ij})_{n \times n}$, the following condition

$$-\min_{1 \leq i \leq n} \frac{c_i}{k_i} + \frac{1}{2} \lambda_{\max} \left[(A + \Omega) + (A + \Omega)^T \right] + \frac{1}{2} + \frac{\|B\|^2}{2(1-\sigma)} < 0 \quad (21)$$

is satisfied, then the controlled uncertain slave system (3) will globally synchronize with the master system (2). Moreover,

$$\lim_{t \rightarrow \infty} (\tilde{c}_i - c_i) = \lim_{t \rightarrow \infty} (\tilde{a}_{ij} - a_{ij}) = \lim_{t \rightarrow \infty} (\tilde{b}_{ij} - b_{ij}) = 0,$$

for all $i, j = 1, 2, \dots, n$.

Proof. Consider the following Lyapunov–Krasovskii functional of the form

$$\begin{aligned} V(t) = & \sum_{i=1}^n \int_0^{e_i(t)} g_i(s) ds + \frac{\|B\|^2}{2(1-\sigma)} \int_{t-\tau(t)}^t g^T(e(s))g(e(s))ds \\ & + \sum_{i=1}^n \left[\frac{1}{\gamma_i} (\tilde{c}_i - c_i)^2 + \sum_{j=1}^n \frac{1}{\alpha_{ij}} (\tilde{a}_{ij} - a_{ij})^2 + \sum_{j=1}^n \frac{1}{\beta_{ij}} (\tilde{b}_{ij} - b_{ij})^2 \right]. \end{aligned} \quad (22)$$

Calculating the derivative of $V(t)$ along the trajectories of the error system (7), we derive that

$$\begin{aligned} \dot{V}(t) = & \sum_{i=1}^n g_i(e_i(t)) \left(-c_i e_i(t) + \sum_{j=1}^n (a_{ij} + w_{ij}) g_j(e_j(t)) + \sum_{j=1}^n b_{ij} g_j(e_j(t - \tau(t))) \right. \\ & \left. - (\tilde{c}_i - c_i) y_i(t) + \sum_{j=1}^n (\tilde{a}_{ij} - a_{ij}) f_j(y_j(t)) + \sum_{j=1}^n (\tilde{b}_{ij} - b_{ij}) f_j(y_j(t - \tau(t))) \right) \\ & + \frac{\|B\|^2}{2(1-\sigma)} \left(g^T(e(t))g(e(t)) - (1 - \dot{\tau}(t)) g^T(e(t - \tau(t)))g(e(t - \tau(t))) \right) \\ & + \sum_{i=1}^n \left[\frac{1}{\gamma_i} (\tilde{c}_i - c_i) \tilde{c}_i + \sum_{j=1}^n \frac{1}{\alpha_{ij}} (\tilde{a}_{ij} - a_{ij}) \tilde{a}_{ij} + \sum_{j=1}^n \frac{1}{\beta_{ij}} (\tilde{b}_{ij} - b_{ij}) \tilde{b}_{ij} \right] \\ \leq & - \sum_{i=1}^n g_i(e_i(t)) c_i e_i(t) + \sum_{i=1}^n \sum_{j=1}^n g_i(e_i(t)) (a_{ij} + w_{ij}) g_j(e_j(t)) \\ & + \sum_{i=1}^n \sum_{j=1}^n g_i(e_i(t)) b_{ij} g_j(e_j(t - \tau(t))) - \sum_{i=1}^n \left[g_i(e_i(t)) (\tilde{c}_i - c_i) y_i(t) \right. \\ & \left. + \sum_{j=1}^n g_i(e_i(t)) (\tilde{a}_{ij} - a_{ij}) f_j(y_j(t)) + \sum_{j=1}^n g_i(e_i(t)) (\tilde{b}_{ij} - b_{ij}) f_j(y_j(t - \tau(t))) \right] \\ & + \frac{\|B\|^2}{2(1-\sigma)} g^T(e(t))g(e(t)) - \frac{\|B\|^2}{2} \left(g^T(e(t - \tau(t)))g(e(t - \tau(t))) \right) \\ & + \sum_{i=1}^n \left[(\tilde{c}_i - c_i) g_i(e_i(t)) y_i(t) - \sum_{j=1}^n (\tilde{a}_{ij} - a_{ij}) g_i(e_i(t)) f_j(y_j(t)) \right. \\ & \left. - \sum_{j=1}^n (\tilde{b}_{ij} - b_{ij}) g_i(e_i(t)) f_j(y_j(t - \tau(t))) \right] \\ \leq & - \min_{1 \leq i \leq n} \frac{c_i}{k_i} g^T(e(t))g(e(t)) + g^T(e(t))(A + \Omega)g(e(t)) \\ & + g^T(e(t))Bg(e(t - \tau(t))) + \frac{\|B\|^2}{2(1-\sigma)} g^T(e(t))g(e(t)) - \frac{\|B\|^2}{2} \left(g^T(e(t - \tau(t)))g(e(t - \tau(t))) \right) \end{aligned}$$

$$\leq \left\{ -\min_{1 \leq i \leq n} \frac{c_i}{k_i} + \frac{1}{2} \lambda_{\max} \left[(A + \Omega) + (A + \Omega)^T \right] + \frac{1}{2} + \frac{\|B\|^2}{2(1-\sigma)} \right\} g^T(e(t))g(e(t)). \quad (23)$$

In the above inequality we use the fact that $g_i(e_i(t))c_i e_i(t) \geq \frac{c_i}{k_i} g_i^2(e_i(t))$. If (22) is satisfied, we can find that $\dot{V}(t) < 0$, and thus $e(t) \rightarrow 0$ as $t \rightarrow \infty$. The global synchronization between (2) and (3) is proved. \square

4. Synchronization of neural networks via state coupling

In this section, by combining the dynamical error feedback theory, the invariance principle of functional differential equations and the adaptive control, a simple, analytical, and rigorous adaptive feedback scheme is proposed for the robust synchronization of almost all kinds of coupled identical neural networks with time-varying delay based on the parameter identification of uncertain chaotic delayed neural networks.

Theorem 4. Under the assumptions (A1) and (A2), let the controller $U(t) = \epsilon \circ (y(t) - x(t)) = \epsilon \circ e(t)$, where the coupling strength $\epsilon = (\epsilon_1, \dots, \epsilon_n)^T \in \mathbf{R}^n$ with the following update law:

$$\dot{\epsilon}_i = -\delta_i e_i^2 \quad (24)$$

and the mark \circ is defined as $\epsilon \circ (y(t) - x(t)) \triangleq [\epsilon_1 e_1(t), \dots, \epsilon_n e_n(t)]^T$. The parameters' adaptive laws of $\tilde{C} = \text{diag}(\tilde{c}_1, \dots, \tilde{c}_n)$, $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ and $\tilde{B} = (\tilde{b}_{ij})_{n \times n}$ are chosen as below:

$$\begin{cases} \dot{\tilde{c}}_i = \gamma_i e_i(t) y_i(t), & i = 1, 2, \dots, n, \\ \dot{\tilde{a}}_{ij} = -\alpha_{ij} e_i(t) f_j(y_j(t)), & i, j = 1, 2, \dots, n, \\ \dot{\tilde{b}}_{ij} = -\beta_{ij} e_i(t) f_j(y_j(t - \tau(t))), & i, j = 1, 2, \dots, n, \end{cases} \quad (25)$$

in which $\delta_i > 0$, $\gamma_i > 0$ ($i = 1, 2, \dots, n$) and $\alpha_{ij} > 0$, $\beta_{ij} > 0$, ($i, j = 1, 2, \dots, n$) are arbitrary constants. Then the controlled uncertain slave system (3) will globally synchronize with the master system (2). Moreover,

$$\lim_{t \rightarrow \infty} (\tilde{c}_i - c_i) = \lim_{t \rightarrow \infty} (\tilde{a}_{ij} - a_{ij}) = \lim_{t \rightarrow \infty} (\tilde{b}_{ij} - b_{ij}) = 0,$$

for all $i, j = 1, 2, \dots, n$.

Proof. Substituting $U(t) = \epsilon \circ (y(t) - x(t))$ into (3), and let $e(t) = y(t) - x(t)$ be the synchronization error between the master system (2) and the controlled slave system (3), one can get the error dynamical system as follows:

$$\begin{aligned} \dot{e}(t) = & -Ce(t) + Ag(e(t)) + Bg(e(t - \tau(t))) - (\tilde{C} - C)y(t) \\ & + (\tilde{A} - A)f(y(t)) + (\tilde{B} - B)f(y(t - \tau(t))) + \epsilon \circ e(t), \end{aligned} \quad (26)$$

where

$$\begin{aligned} e(t) &= (e_1(t), \dots, e_n(t))^T, \\ f(y(t)) &= (f(y_1(t)), \dots, f(y_n(t)))^T, \\ g(e(t)) &= (g(e_1(t)), \dots, g(e_n(t)))^T \text{ with } g_i(e_i(t)) = f_i(e_i(t) + x_i(t)) - f_i(x_i(t)), i = 1, 2, \dots, n. \end{aligned}$$

For the error dynamical system (27), we design the following Lyapunov functional:

$$\begin{aligned} V(t) = & \frac{1}{2} \left\{ e^T(t)e(t) + \sum_{i=1}^n \frac{1}{\delta_i} (\epsilon_i + l)^2 + \frac{1}{1-\sigma} \int_{t-\tau(t)}^t g^T(e(s))g(e(s))ds \right. \\ & \left. + \sum_{i=1}^n \left[\frac{1}{\gamma_i} (\tilde{c}_i - c_i)^2 + \sum_{j=1}^n \frac{1}{\alpha_{ij}} (\tilde{a}_{ij} - a_{ij})^2 + \sum_{j=1}^n \frac{1}{\beta_{ij}} (\tilde{b}_{ij} - b_{ij})^2 \right] \right\}, \end{aligned} \quad (27)$$

where l is a positive constant to be determined.

Calculating the derivative of (28) along the trajectories of the error system (27), we have

$$\dot{V}(t) = e^T(t)\dot{e}(t) - \sum_{i=1}^n (\epsilon_i + l)e_i^2(t) + \frac{1}{2(1-\sigma)} g^T(e(t))g(e(t)) - \frac{1-\dot{\tau}(t)}{2(1-\sigma)} g^T(e(t - \tau(t)))g(e(t - \tau(t)))$$

$$\begin{aligned}
& + \sum_{i=1}^n \left[\frac{1}{\gamma_i} (\tilde{c}_i - c_i) \tilde{c}_i + \sum_{j=1}^n \frac{1}{\alpha_{ij}} (\tilde{a}_{ij} - a_{ij}) \tilde{a}_{ij} + \sum_{j=1}^n \frac{1}{\beta_{ij}} (\tilde{b}_{ij} - b_{ij}) \tilde{b}_{ij} \right] \\
& = e^T(t) (-Ce(t) + Ag(e(t)) + Bg(e(t - \tau(t))) - (\tilde{C} - C)y(t) \\
& \quad + (\tilde{A} - A)f(y(t)) + (\tilde{B} - B)f(y(t - \tau(t))) + \epsilon \circ e(t)) - \sum_{i=1}^n (\epsilon_i + l)e_i^2(t) \\
& \quad + \frac{1}{2(1-\sigma)} g^T(e(t))g(e(t)) - \frac{1-\dot{\tau}(t)}{2(1-\sigma)} g^T(e(t - \tau(t)))g(e(t - \tau(t))) \\
& \quad + \sum_{i=1}^n \left[\frac{1}{\gamma_i} (\tilde{c}_i - c_i) \gamma_i e_i(t) y_i(t) - \sum_{j=1}^n \frac{1}{\alpha_{ij}} (\tilde{a}_{ij} - a_{ij}) \alpha_{ij} e_i(t) f(y_i(t)) \right. \\
& \quad \left. - \sum_{j=1}^n \frac{1}{\beta_{ij}} (\tilde{b}_{ij} - b_{ij}) \beta_{ij} e_i(t) f(y_i(t - \tau(t))) \right] \\
& \leq -e^T(t)Ce(t) + e^T(t)Ag(e(t)) + e^T(t)Bg(e(t - \tau(t))) - le^T(t)e(t) \\
& \quad + \frac{1}{2(1-\sigma)} g^T(e(t))g(e(t)) - \frac{1-\dot{\tau}(t)}{2(1-\sigma)} g^T(e(t - \tau(t)))g(e(t - \tau(t))). \tag{28}
\end{aligned}$$

Recalling (A2): $0 \leq \dot{\tau}(t) \leq \sigma < 1$, one can get $-\frac{1-\dot{\tau}(t)}{2(1-\sigma)} \leq -1$.

By Lemma 1, and taking $\Sigma_1, \Sigma_2, \Sigma_3$ as the corresponding identity matrices, we obtain

$$\begin{aligned}
\dot{V}(t) & \leq -e^T(t)Ce(t) + \frac{1}{2}e^T(t)AA^Te(t) + \frac{1}{2}g^T(e(t))g(e(t)) \\
& \quad + \frac{1}{2}e^T(t)BB^Te(t) + \frac{1}{2}g^T(e(t - \tau(t)))g(e(t - \tau(t))) - le^T(t)e(t) \\
& \quad + \frac{1}{2(1-\sigma)} g^T(e(t))g(e(t)) - \frac{1}{2}g^T(e(t - \tau(t)))g(e(t - \tau(t))). \tag{29}
\end{aligned}$$

Making use of (9), we can obtain

$$g^T(e(t))g(e(t)) \leq ke^T(t)e(t), \tag{30}$$

where $k = \max\{k_i^2 | i = 1, 2, \dots, n\}$.

Substituting (31) into the right-hand side of inequality (30) yields

$$\begin{aligned}
\dot{V}(t) & \leq -e^T(t)Ce(t) + \frac{1}{2}e^T(t)AA^Te(t) + \frac{1}{2}e^T(t)BB^Te(t) + \left(\frac{1}{2}k + \frac{1}{2(1-\sigma)}k - l \right) e^T(t)e(t) \\
& \leq e^T(t) \left[\lambda_{\max}(-C) + \lambda_{\max} \left(\frac{1}{2}AA^T \right) + \lambda_{\max} \left(\frac{1}{2}BB^T \right) + \frac{1}{2}k + \frac{1}{2(1-\sigma)}k - l \right] e(t). \tag{31}
\end{aligned}$$

The constant l can be properly chosen as

$$l = \lambda_{\max}(-C) + \lambda_{\max} \left(\frac{1}{2}AA^T \right) + \lambda_{\max} \left(\frac{1}{2}BB^T \right) + \frac{1}{2}k + \frac{1}{2(1-\sigma)}k + 1,$$

then one can get $\dot{V} \leq -e^T(t)e(t)$.

It is obvious that $\dot{V} = 0$ if and only if $e(t) = 0$. According to the well-known invariant principle of functional differential equations, the orbit of system (27), starting with arbitrary initial value, converges asymptotically to the largest invariant set E contained in $\dot{V}(t) = 0$ as $t \rightarrow \infty$, where the set $E = \{e(t) = 0 | \tilde{C} = C, \tilde{A} = A, \tilde{B} = B, \epsilon = \epsilon_0 \in \mathbf{R}^n\}$. Then, the synchronization of coupled neural networks (2) and (3) with time-varying delay is achieved under the dynamical coupling (25). This completes the proof. \square

Similar to Theorem 3, when the two chaotic neural networks are connected by state coupling, we have

Theorem 5. Under the assumptions (A1) and (A2), let the controller $u_i(t) = \sum_{j=1}^n w_{ij} (y_j(t) - x_j(t))$ and the parameters' adaptive laws of \tilde{C} , \tilde{A} and \tilde{B} are chosen as below:

$$\begin{cases} \dot{\tilde{c}}_i = \gamma_i g_i(e_i(t)) y_i(t), & i = 1, 2, \dots, n, \\ \dot{\tilde{a}}_{ij} = -\alpha_{ij} g_i(e_i(t)) f_j(y_j(t)), & i, j = 1, 2, \dots, n, \\ \dot{\tilde{b}}_{ij} = -\beta_{ij} g_i(e_i(t)) f_j(y_j(t - \tau(t))), & i, j = 1, 2, \dots, n, \end{cases} \quad (32)$$

in which $\gamma_i > 0$ ($i = 1, 2, \dots, n$) and $\alpha_{ij} > 0$, $\beta_{ij} > 0$, ($i, j = 1, 2, \dots, n$) are arbitrary constants. If, provided that for controller gain matrix $\Omega = (w_{ij})_{n \times n}$, the following condition

$$\frac{1}{2} \lambda_{\max} [\Omega + \Omega^T] < \min_{1 \leq i \leq n} \frac{c_i}{k_i} - \frac{1}{2} \lambda_{\max} [A + A^T] - \frac{\|B\|^2}{2} - \frac{\|B\|^2}{2(1 - \sigma)} \quad (33)$$

is satisfied, then the controlled uncertain slave system (3) will globally synchronize with the master system (2). Moreover,

$$\lim_{t \rightarrow \infty} (\tilde{c}_i - c_i) = \lim_{t \rightarrow \infty} (\tilde{a}_{ij} - a_{ij}) = \lim_{t \rightarrow \infty} (\tilde{b}_{ij} - b_{ij}) = 0,$$

for all $i, j = 1, 2, \dots, n$.

The proof of this theorem is similar to that of Theorem 3, except that the first term of the Lyapunov functional is defined as $e^T(t)e(t)$ and the second term is changed accordingly.

For the coupled neural networks without time-varying delay [i.e., $B = 0$ in (2) and \tilde{B} in (3)], one can easily derive the following corollary. Consider the following two coupled neural networks without delay (master system and slave system):

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + J, \quad (34)$$

and

$$\dot{y}(t) = -\tilde{C}y(t) + \tilde{A}f(y(t)) + J + \epsilon \circ (y(t) - x(t)). \quad (35)$$

Corollary 1. Under the assumptions (A1) and (A2), the coupled neural networks (35) and (36) can be synchronized when the coupling strength $\epsilon = (\epsilon_1, \dots, \epsilon_n)^T$ is updated by the following law:

$$\dot{\epsilon}_i = -\delta_i \epsilon_i^2, \quad (36)$$

where $\delta_i > 0$ ($i = 1, 2, \dots, n$) are arbitrary positive constants and the parameters' adaptive laws of $\tilde{C} = \text{diag}(\tilde{c}_1, \dots, \tilde{c}_n)$, $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ and $\tilde{B} = (\tilde{b}_{ij})_{n \times n}$ are chosen as below:

$$\begin{cases} \dot{\tilde{c}}_i = \gamma_i e_i(t) y_i(t), & i = 1, 2, \dots, n, \\ \dot{\tilde{a}}_{ij} = -\alpha_{ij} e_i(t) f_j(y_j(t)), & i, j = 1, 2, \dots, n, \\ \dot{\tilde{b}}_{ij} = -\beta_{ij} e_i(t) f_j(y_j(t - \tau(t))), & i, j = 1, 2, \dots, n, \end{cases} \quad (37)$$

in which $\gamma_i > 0$ ($i = 1, 2, \dots, n$) and $\alpha_{ij} > 0$, $\beta_{ij} > 0$, ($i, j = 1, 2, \dots, n$) are arbitrary constants. Moreover,

$$\lim_{t \rightarrow \infty} (\tilde{c}_i - c_i) = \lim_{t \rightarrow \infty} (\tilde{a}_{ij} - a_{ij}) = \lim_{t \rightarrow \infty} (\tilde{b}_{ij} - b_{ij}) = 0,$$

for all $i, j = 1, 2, \dots, n$.

Remark 1. Similar to the analysis in [35], from Theorem 4, one can easily see that, the constant δ_i can be chosen properly to adjust the synchronization speed. A sufficiently large adaptive gain δ_i would lead to fast synchronization, while for sufficiently small adaptive gain δ_i , the time to achieve synchronization may be quite long. Also, this method is quite robust against the effect of noise and the condition satisfying this scheme is also very loose, i.e., only choosing the constant l properly.

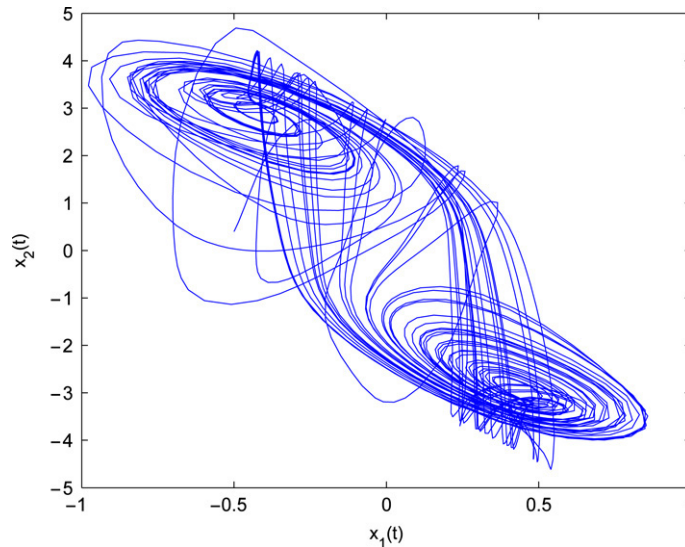


Fig. 1. Attractor of neural network model (39).

Remark 2. As stated in [28] these estimation approaches can respond dynamically to changes in identifying parameters of the master system. It is useful to point out that the distinguished characteristics of our method can be applied to almost all the chaotic delayed neural networks with the uniform Lipschitz activation functions. Therefore, the approaches developed here are very convenient to implement in practice.

Comparisons. (1) In most of the previous literature, the usual linear feedback scheme with a fixed coupling strength or a fixed feedback matrix is used regardless of the starting initial values; thus, the strength must be maximum, which is a kind of waste in practice. (2) In Refs. [35–37,39,40], all the results are based on the coupled neural networks with certain parameters. So our results have more expansive application foreground. Moreover, using both adaptive feedback scheme and output or state coupling method, the criteria obtained here improve and extend the results reported in Refs. [35–40].

5. Numerical examples

In this section, as an application of the above-derived theoretical criteria, some numerical examples are shown for the robust synchronization and identification problem of two coupled chaotic neural networks.

Example 1. First consider the following two-order Hopfield neural networks with time-varying delay:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) + J, \quad (38)$$

with

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2.1 & -0.12 \\ -5.1 & 3.2 \end{bmatrix}, \quad B = \begin{bmatrix} -1.6 & -0.1 \\ -0.2 & -2.4 \end{bmatrix}, \quad J = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$\tau(t) = e^t / (1 + e^t), \text{ where } x(t) = [x_1(t), x_2(t)]^T, \quad f(x(t)) = [\tanh(x_1(t)), \tanh(x_2(t))]^T.$$

So, we can see that $k_1 = k_2 = 1$, so $K = 1$. Moreover,

$$\tau^* = 1, \quad \dot{\tau}(t) = \frac{e^t}{(1 + e^t)^2} \in [0, 0.5],$$

i.e., $\sigma = 0.5$. Obviously, the assumptions (A1) and (A2) hold.

It should be noted that this neural network is actually a chaotic delayed Hopfield neural networks (see, Fig. 1 with initial values $\phi_1(s) = -0.5$, $\phi_2(s) = 0.4$, $\forall s \in [-1, 0]$).

In order to verify the effectiveness of the proposed method, let the master output signals be from the delayed neural networks (39). For simplicity, we assume only that the four parameters $a_{11} = 2.1$, $a_{22} = 3.2$, $b_{11} = -1.6$ and $b_{22} = -2.4$ will be identified, then the controlled slave system is given by the following equation:

$$\dot{y}(t) = -\tilde{C}y(t) + \tilde{A}f(y(t)) + \tilde{B}f(y(t - \tau(t))) + J + U(t), \quad (39)$$

where $y(t) = (y_1(t), y_2(t))^T$ and

$$\tilde{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{a}_{11} & -0.12 \\ -5.1 & \tilde{a}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{b}_{11} & -0.1 \\ -0.2 & \tilde{b}_{22} \end{bmatrix}, \quad J = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and the controller is chosen as $U(t) = \Omega(f(y(t)) - f(x(t)))$ with the output coupling controller gain matrix

$$\Omega = \begin{bmatrix} -12 & 1 \\ 5 & -14 \end{bmatrix}.$$

According to Theorem 3, one can easily construct the parameters' adaptive laws as follows:

$$\begin{cases} \dot{\tilde{a}}_{11} = -5.2 [\tanh(y_1(t)) - \tanh(x_1(t))] \tanh(y_1(t)), \\ \dot{\tilde{a}}_{22} = -8.7 [\tanh(y_2(t)) - \tanh(x_2(t))] \tanh(y_2(t)), \\ \dot{\tilde{b}}_{11} = -2.6 [\tanh(y_1(t)) - \tanh(x_1(t))] \tanh(y_1(t)), \\ \dot{\tilde{b}}_{22} = -9.2 [\tanh(y_2(t)) - \tanh(x_2(t))] \tanh(y_2(t)). \end{cases} \quad (40)$$

It is easy to check that

$$\lambda_{\max}(-C) + \lambda_{\max}\left(\frac{1}{2}(A + \Omega)(A + \Omega)^T\right) + \lambda_{\max}\left(\frac{1}{2}BB^T\right) + \frac{1}{2}k + \frac{1}{2(1-\sigma)}k = 63.4759 > 0, \quad (41)$$

so condition (6) in Theorem 1 is not satisfied. However, if we let $p_1 = p_2 = 1$, we can see that

$$\begin{aligned} & -p_1c_1 + p_1(a_{11} + w_{11})k_1 + p_1|a_{12} + w_{12}|k_2 + p_2|a_{21} + w_{21}|k_1 \\ & + \sum_{j=1}^2 p_1|b_{1j}|k_j + \frac{1}{1-\sigma} \sum_{j=1}^2 p_1|b_{j1}|k_1 = -4.6200 < 0, \end{aligned} \quad (42)$$

$$\begin{aligned} & -p_2c_2 + p_2(a_{22} + w_{22})k_2 + p_2|a_{21} + w_{21}|k_1 + p_1|a_{12} + w_{12}|k_2 \\ & + \sum_{j=1}^2 p_2|b_{2j}|k_j + \frac{1}{1-\sigma} \sum_{j=1}^2 p_2|b_{j2}|k_2 = -3.2200 < 0, \end{aligned} \quad (43)$$

and

$$-\min_{1 \leq i \leq n} \frac{c_i}{k_i} + \frac{1}{2} \lambda_{\max} \left[(A + \Omega) + (A + \Omega)^T \right] + \frac{\|B\|^2}{2} + \frac{\|B\|^2}{2(1-\sigma)} = -4.4602 < 0. \quad (44)$$

Therefore, both the condition (13) in Theorem 2 and the condition (22) in Theorem 3 are satisfied, i.e., from Theorems 2 and 3, one can conclude that the controlled uncertain slave system (40) is globally synchronous with the master system (39) and satisfies

$$\lim_{t \rightarrow \infty} (\tilde{a}_{ii} - a_{ii}) = \lim_{t \rightarrow \infty} (\tilde{b}_{ii} - b_{ii}) = 0, \quad i = 1, 2.$$

Let the initial conditions of the state variables and the unknown parameters of the controlled slave system be as follows:

$$\begin{aligned} & (\phi_1(s), \phi_2(s))^T = (0.2, 0.5)^T, \quad (\psi_1(s), \psi_2(s))^T = (-1.3, 2.1)^T, \quad s \in [-1, 0], \\ & (\tilde{a}_{11}(0), \tilde{a}_{22}(0), \tilde{b}_{11}(0), \tilde{b}_{22}(0))^T = (0.2, -0.5, -2, 1.2)^T. \end{aligned}$$

The numerical simulation shows that parameter identification and the two coupled neural networks' synchronization are achieved successfully (see Figs. 2–4).

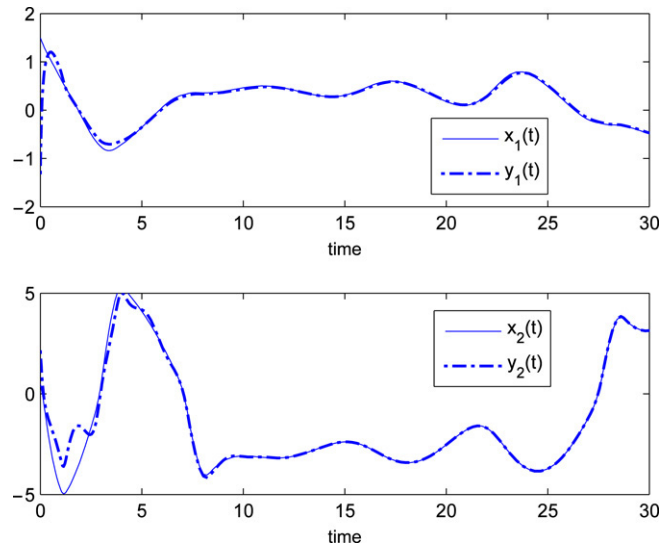


Fig. 2. The temporal evolution of each variable of the coupled neural networks (39) and (40).

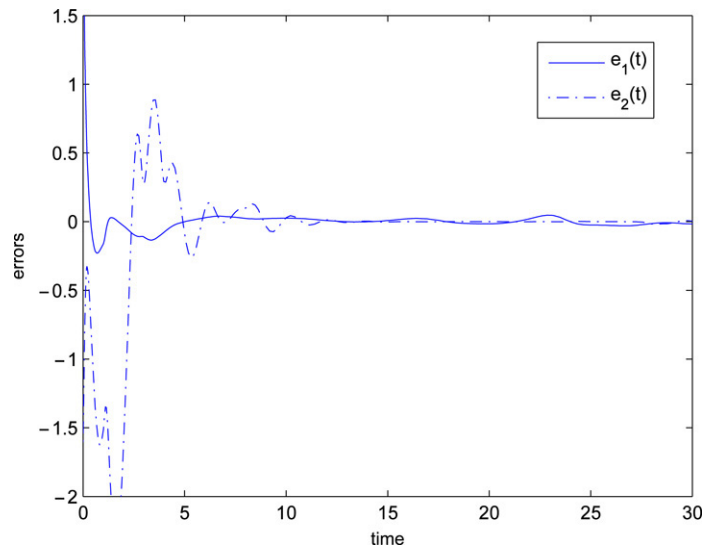


Fig. 3. The plot of synchronization errors of the coupled neural networks (39) and (40).

Example 2. Consider the two coupled neural networks (39) and (40) again. Here we will achieve the robust synchronization of the coupled neural networks with time-varying delay via state coupling and adaptive feedback scheme. First, the controller is chosen as $U(t) = \epsilon \circ (y(t) - x(t))$ with the feedback strength $\epsilon = \text{diag}(\epsilon_1, \epsilon_2)$, then the controlled slave system is given by the following equation:

$$\dot{y}(t) = -\tilde{C}y(t) + \tilde{A}f(y(t)) + \tilde{B}f(y(t - \tau(t))) + J + \epsilon \circ (y(t) - x(t)), \quad (45)$$

where $y(t) = (y_1(t), y_2(t))^T$ and

$$\tilde{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{a}_{11} & -0.12 \\ -5.1 & \tilde{a}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{b}_{11} & -0.1 \\ -0.2 & \tilde{b}_{22} \end{bmatrix}, \quad J = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

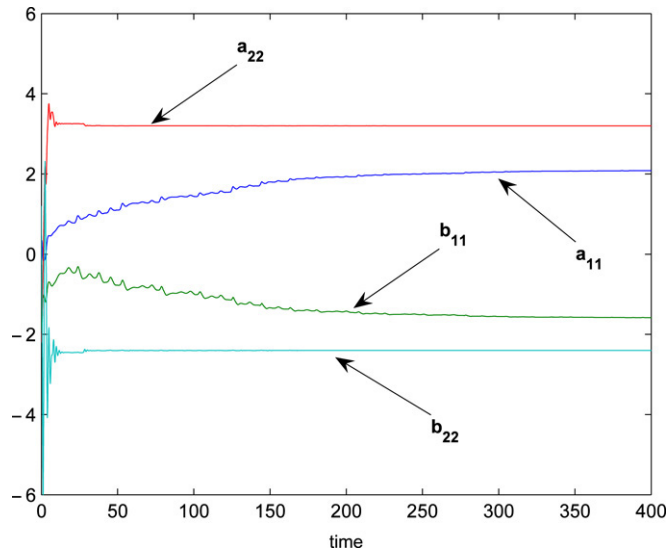


Fig. 4. Parameter identification of the coupled neural networks (39) and (40).

According to Theorem 4, one can easily construct the feedback strength update laws and the parameters' adaptive laws as follows:

$$\begin{cases} \dot{\epsilon}_1 = -0.7(y(1) - x(1))^2, \\ \dot{\epsilon}_2 = -0.7(y(2) - x(2))^2, \\ \dot{\tilde{a}}_{11} = -4.1(y_1(t) - x_1(t)) \tanh(y_1(t)), \\ \dot{\tilde{a}}_{22} = -6.0(y_2(t) - x_2(t)) \tanh(y_2(t)), \\ \dot{\tilde{b}}_{11} = -3.1(y_1(t) - x_1(t)) \tanh(y_1(t)), \\ \dot{\tilde{b}}_{22} = -3.5(y_2(t) - x_2(t)) \tanh(y_2(t)). \end{cases} \quad (46)$$

As we can see, the adaptive gains are $\delta_1 = \delta_2 = 0.7$.

In the following numerical simulations, we take all the initial conditions as

$$\begin{aligned} (\phi_1(s), \phi_2(s))^T &= (0.2, 0.5)^T, & (\psi_1(s), \psi_2(s))^T &= (-1.3, 2.1)^T, & s \in [-1, 0], \\ (\tilde{a}_{11}(0), \tilde{a}_{22}(0), \tilde{b}_{11}(0), \tilde{b}_{22}(0))^T &= (0.2, -0.5, -2, 1.2)^T, \end{aligned}$$

respectively, and $\epsilon_1(0) = \epsilon_2(0) = 0$. The corresponding simulation results are shown in Figs. 5–9. Fig. 5 shows the temporal evolution of each variable and the dynamical coupling strength ϵ_1, ϵ_2 in Example 2, when the adaptive gains are $\delta_1 = \delta_2 = 0.7$. The corresponding simulation results for parameter identification and synchronization errors are shown in Fig. 6 and Fig. 7, respectively. When let $\delta_1 = \delta_2 = 1.5$, namely, increasing the update gain of coupling strength, the numerical simulation results are given in Fig. 8, and we find that the time to achieve synchronization is shorter. Let $\delta_1 = 0, \delta_2 = 0.7$, namely only x_2 is chosen as the drive signal, and the corresponding simulations are shown in Fig. 9, while in Fig. 10, we set $\delta_1 = 0.7, \delta_2 = 0$, namely only x_1 is chosen as the drive signal. From Figs. 9 and 10, we can see that the coupling of only variables x_1 and y_1 can drive the two coupled neural networks (39) and (40) synchronized, while the coupling of only variables x_2 and y_2 cannot.

6. Conclusions

In this paper, based on parameter identification and via output or state coupling, the global synchronization between two coupled neural networks with time-varying delay are studied. Practical and less restrictive conditions are presented for the delayed neural networks. The numerical simulations validate the effectiveness of the proposed methods. As we know, how to achieve synchronization is a key requirement in the design of chaos-based secure

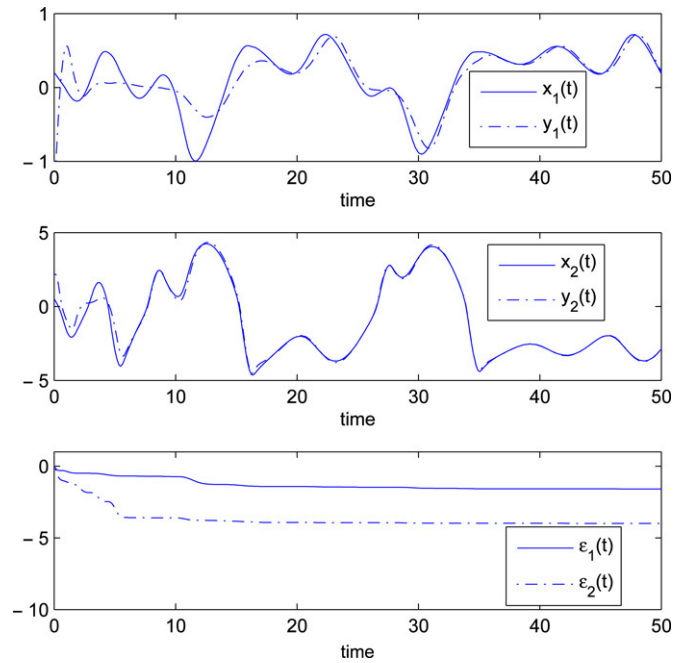


Fig. 5. The temporal evolution of each variable and the dynamical coupling strength ϵ_1, ϵ_2 in Example 2, when the adaptive gains are $\delta_1 = \delta_2 = 0.7$.

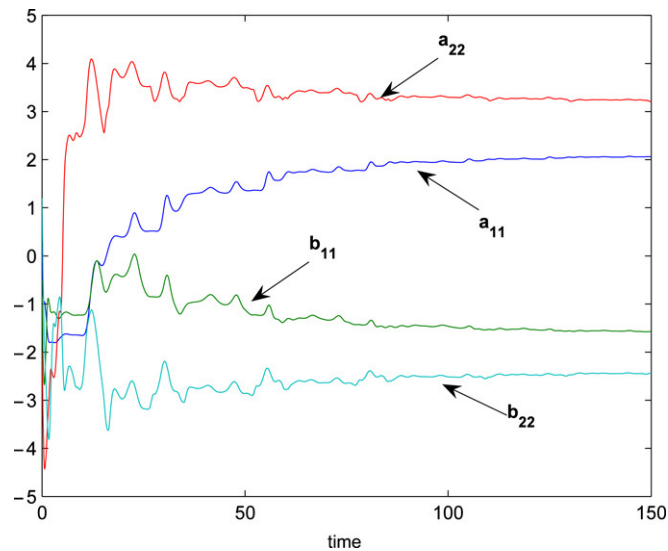


Fig. 6. Parameter identification in Example 2 with the parameters' adaptive laws (47).

communication schemes. A possible application of the proposed methods is to secure message transmission using parameter modulation. Moreover, many other synchronization methods are valid for chaotic systems only when the systems' parameters are known. But in many practical situations, the values of some systems' parameters are not exactly known. Therefore, the scheme described in this paper which can be used to identify the parameters of chaotic systems has the potential application in estimating system's parameters. It is believed that the results should provide some practical guidelines for chaos communication in engineering applications.

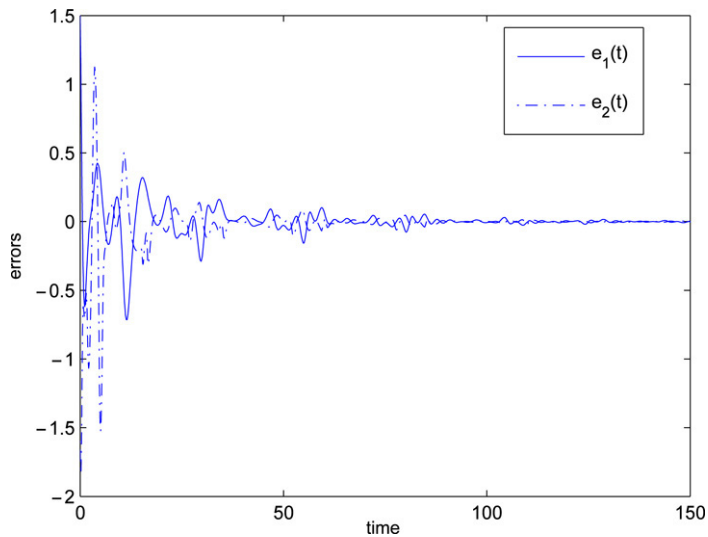


Fig. 7. The plot of synchronization errors in Example 2, when the adaptive gains are $\delta_1 = \delta_2 = 0.7$.

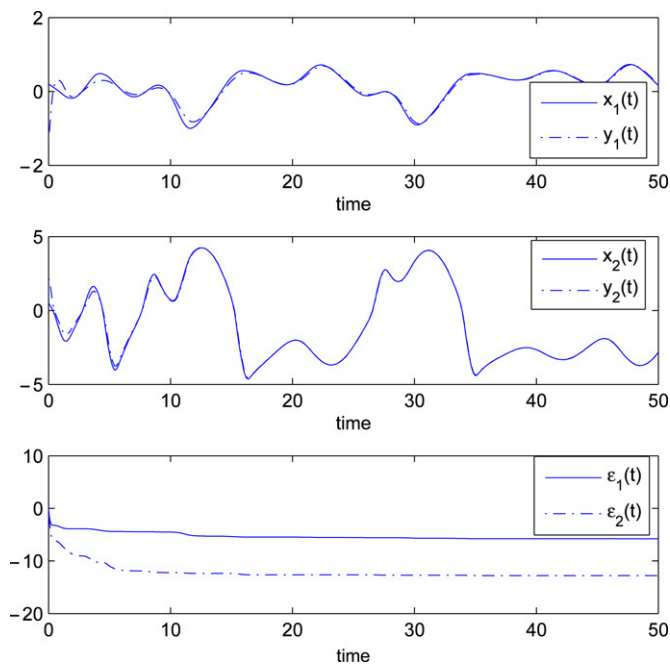


Fig. 8. The temporal evolution of each variable and the dynamical coupling strength ϵ_1, ϵ_2 in Example 2, when the adaptive gains are $\delta_1 = \delta_2 = 1.5$.

Acknowledgements

The authors thank the Associate Editor and the anonymous reviewers for their careful reading of this paper and constructive comments. This work was supported by NSFC (No. 60674026), the Key Research Foundation of Science and Technology of the Ministry of Education of China (No. 107058), the National Natural Science Foundation of Jiangsu Province (No. BK2007016), the Jiangsu Provincial Program for Postgraduate Scientific Innovative Research of Jiangnan University (No. CX07B_116z), and PIRTJiangnan.

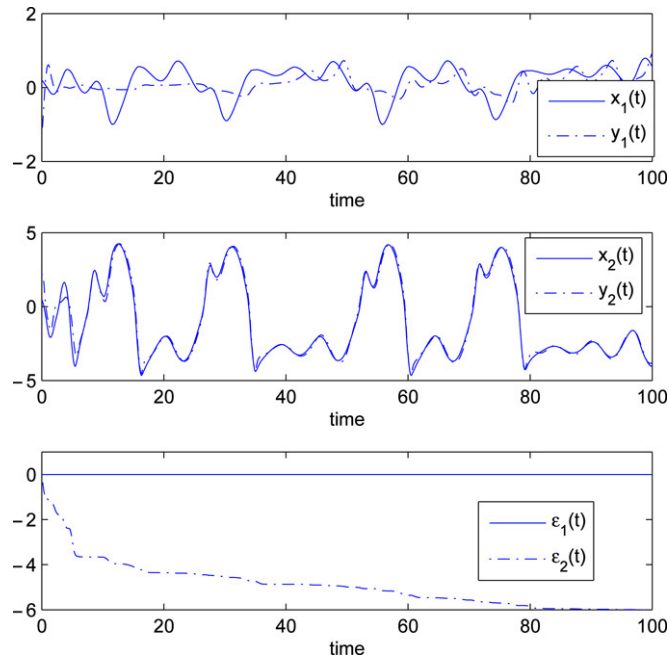


Fig. 9. The temporal evolution of each variable and the dynamical coupling strength ϵ_1, ϵ_2 driven only by the signal x_2 , and the synchronization cannot be achieved in Example 2.

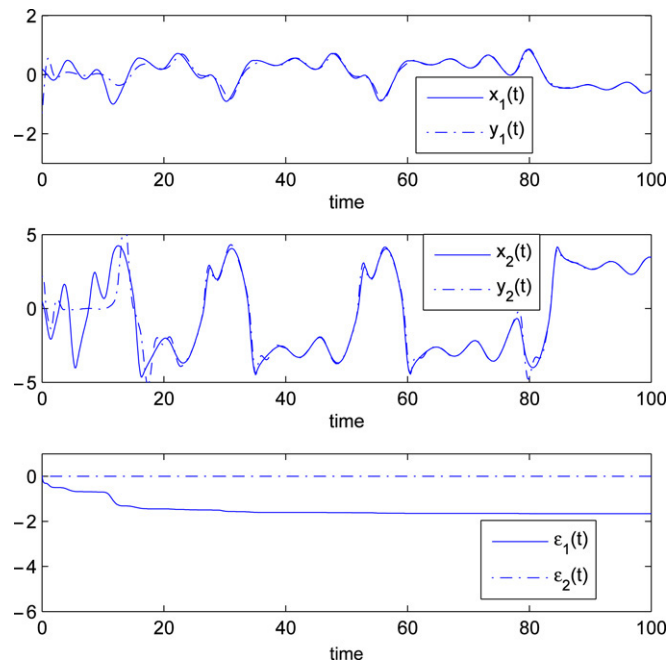


Fig. 10. The temporal evolution of each variable and the dynamical coupling strength ϵ_1, ϵ_2 driven only by the signal x_1 , and the synchronization is achieved in Example 2.

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